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PARTIAL DIFFERENTIAL EQUATIONS

Boundary Value Problem for a Second-Order Elliptic Equation in the Exterior of an Ellipse

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Abstract—We consider a boundary value problem for a second-order linear elliptic differential equation with constant coefficients in a domain that is the exterior of an ellipse. The boundary conditions of the problem contain the values of the function itself and its normal derivative. We give a constructive solution of the problem and find the number of solvability conditions for the inhomogeneous problem as well as the number of linearly independent solutions of the homogeneous problem. We prove the boundary uniqueness theorem for the solutions of this equation.

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Let

$$L = \sum_{k=0}^{2} a_k \frac{\partial^2}{\partial x^{2-k} \partial y^k}$$

be an elliptic operator with constant complex coefficients a_0, a_1, and a_2. The ellipticity means that $a_2 \neq 0$ and the characteristic polynomial $a_0 + a_1 s + a_2 s^2$ has no real roots. We denote the roots of this polynomial by s_1 and s_2. In a domain $D \subset \mathbb{C}$, consider the equation

$$LF = 0. \tag{1}$$

The real and imaginary parts of solutions of Eq. (1) satisfy the biharmonic equation (the main equation of the plane isotropic theory of elasticity) if $s_1 = s_2 = i$ (see [1, p. 105]) and the main equation of the plane anisotropic theory of elasticity if $s_1 \neq s_2$, $\operatorname{Im} s_1 > 0$, and $\operatorname{Im} s_2 > 0$ (see [2, p. 32]).

Consider the following boundary value problem.

Let D be the exterior of the ellipse

$$\Gamma: \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with clockwise sense. The problem is to find a solution $F(z)$, $z = x + iy$, of Eq. (1) that is bounded at infinity, is regular in the domain D, has continuous normal derivative on $\Gamma = \partial D$, and satisfies the following boundary conditions on Γ :

$$\operatorname{Re}[a_0(t) - ib_0(t)]F(t) = c_0(t), \qquad \operatorname{Re}[a_1(t) - ib_1(t)]\frac{\partial F(t)}{\partial n} = c_1(t), \qquad t \in \Gamma. \tag{2}$$

Here $a_0(t), b_0(t), a_1(t), b_1(t), c_0(t), c_1(t) \in H_\mu(\Gamma)$, $0 < \mu < 1$, are given real functions such that $[a_0(t)]^2 + [b_0(t)]^2 \neq 0$ and $[a_1(t)]^2 + [b_1(t)]^2 \neq 0$.

For $s_1 = s_2 = i$, Eq. (1) defines bianalytic functions. For these functions, problem (2) was solved in explicit form for a disk and a half-plane (see [3, Sec. 32; 4; 5]); for an arbitrary domain, its qualitative study was carried out in [6, Chap. 4, Sec. 11; 7]. In what follows, we consider the case in which $s_1 \neq s_2$, $\operatorname{Im} s_1 > 0$, and $\operatorname{Im} s_2 > 0$. In this case, problem (2) was solved in [8] in explicit